Prof. Amador Martin-Pizarro — Albert-Ludwigs-Universität Freiburg
Übungen: Xier Ren — Sommersemester 2024

Topology

Problem Sheet 10

Deadline: 2 July 2024, 15h

Exercise 1 (4 Points).

Given a continuous map $f : X \to Y$ of topological spaces with $f(x_0) = y_0$, consider the induced group homomorphism $f_\star : \pi_1(X, x_0) \to \pi_1(Y, y_0)$.

a) If f is injective, does $f_\star$ have to be injective?

b) If f is surjective, does $f_\star$ have to be surjective?

Hint: Use some fundamental groups we already know.

Exercise 2 (5 Points).

Consider the following topology $\mathcal{T} = \{\emptyset, X, \{1\}\}$ on the set $X = \{1, 2\}$.

a) Is X path-connected? Is X connected?

b) Is X simply connected?

Hint: A map is continuous if and only if the preimage of every closed set is closed.

c) Is X contractible? (See Sheet 9, Exercise 1)

Exercise 3 (8 Points).

Set $Y = \{(0,0,0,w)\}_{w\in\mathbb{R}\setminus\mathbb{Z}} \cup \bigcup_{n\in\mathbb{Z}} \{(x,y,z,n) \in \mathbb{R}^4 \mid (x-1)^2 + y^2 + z^2 = 1\}$ and

$$X = \{(x,y,z) \in \mathbb{R}^3 | (x-1)^2 + y^2 + z^2 = 1\} \cup \{(x,y,0) \in \mathbb{R}^3 | (x+1)^2 + y^2 = 1\},$$

each equipped with the subspace topology with respect to the euclidean topology.

a) Show that Y together with the map

$$f : \begin{array}{ccc} Y & \to & X \\ (x,y,z,w) & \mapsto & \begin{cases} (x,y,z) & w \in \mathbb{Z} \\ (\cos(2\pi w) - 1, \sin(2\pi w), 0) & w \notin \mathbb{Z}. \end{cases} \end{array}$$

is a covering of X.

b) Show that Y is simply connected, so Y is a universal cover of X.

Hint: Decompose a loop (up to homotopy) in a product of paths.

c) Is X simply connected?

Exercise 4 (3 Points).

Given the universal cover

$$\pi : \begin{array}{ccc} \mathbb{R} & \to & \mathbb{S}^1 \\ t & \mapsto & (\cos(2\pi t), \sin(2\pi t)) \end{array},$$

show that the map $f : \begin{array}{ccc} \mathbb{S}^1 & \to & \mathbb{S}^1 \\ z & \mapsto & -z \end{array}$ lifts to a map $g : \mathbb{R} \to \mathbb{R}$ with $\pi \circ g = f \circ \pi$.

Is there such a lifting g as above with $g \circ g = \mathrm{Id}_{\mathbb{R}}$?

Die Übungsblätter können zu zweit eingereicht werden. Abgabe der Übungsblätter im entsprechenden Fach im Keller des mathematischen Instituts.